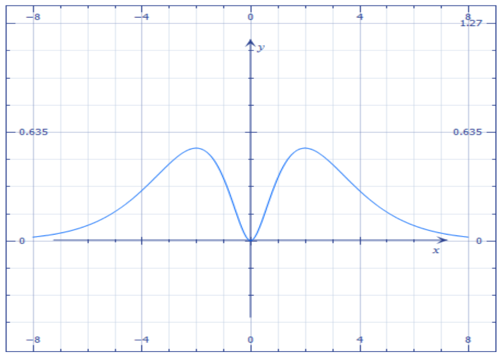
Functions: Even and Odd Functions and Composition

Definitions

Even Function – A function f is even if $f(-x) = f(x)$ for *all* x.

- The graph of an even function is symmetric about x-axis. This means that the graph for negative x is the mirror image of that for positive x.
- Example: $f(x) = x^2e^{-|x|}$

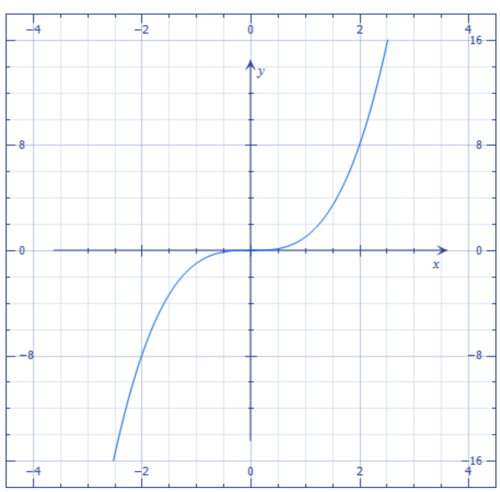


Notice the symmetry in the graph. To prove that $f(x)$is an even function, we must show that if $f(-x) = f(x)$ for all values of x. We do this by calculating $f(-x)$.

$$f(x) = x^2e^{-|x|}$$
$$f(-x) = (-x)^2e^{-|-x|}$$
$$f(-x) = (x)^2e^{-|x|}$$
$$f(-x) = x^2e^{-|x|} = f(x)$$

Odd Function – A function f is odd if $f(-x) = -f(x)$ for *all* x.

- The graph of an odd function is symmetric about the origin.
- Example: $f(x) = x^3$



Notice the symmetry of the function. To prove that this function is odd, calculate $f(-x)$

$$f(x) = x^3$$
$$f(-x) = (-x)^3$$
$$f(-x) = (-1)^3(x)^3$$
$$f(-x) = -1x^3 = -x^3 = -f(x)$$

Even and odd functions do not necessarily follow the same rules with regard to addition, substraction, multiplication and division as do regular numbers. For example, the sum of two odd numbers is even, but the sum of two odd functions is odd. We will prove this.

Prove that the sum of two odd functions is another odd function.

First, let $h(x)$ be the sum of two odd functions. We need to show that $h(x)$ is odd, which, mathematically, means that $h(-x) = -h(x)$.

Now let $h(x) = f(x) + g(x)$, where $f(x)$ and $g(x)$ are arbitrary odd functions. Since f and g are odd they satisfy the following conditions:

$$f(-x) = -f(x) \qquad (1)$$

and

$$g(-x) = -g(x) \qquad (2)$$

Now, let's calculate $h(-x)$:

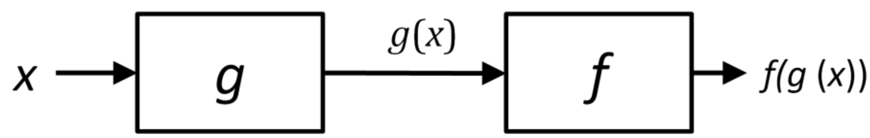
$$h(-x) = f(-x) + g(-x)$$

Using (1) and (2) above to substitute for $f(-x)$ and $g(-x)$ leads to

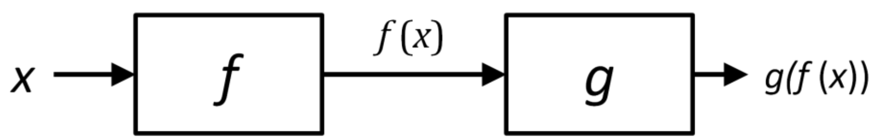
$$h(-x) = -f(x) - g(x)$$
$$= -(f(x) + g(x))$$
$$= -h(x)$$

Note: Functions may be even or odd or neither.

Composition of Functions – A composition of two functions is the application of one function after another. The composition of f and g is denoted by $f(g(x))$ or $g(f(x))$ depending on the order the functions are applied. Graphically, $f(g(x))$ can be pictured as



and $g(f(x))$ can be pictured as



Let $f(x) = \frac{1}{x}$ and $g(x) = 2^x$ then

$$f(g(x)) = f(2^x) = \frac{1}{2^x}$$

and

$$g(f(x)) = g\left(\frac{1}{x}\right) = 2^{\frac{1}{x}}$$

In general, $f(g(x)) \neq g(f(x))$. That is the case in the above example. To show that this is true, it is necessary only to show that $f(g(x)) \neq g(f(x))$ for some value of x. To show that two functions are equal to each other, they must be equal for all values of x. However, if there is some x for which they are unequal, even if it is only one, then the two functions are not equal to each other. So, in the above example, let $x = 1$. Then, $f(g(1)) = \frac{1}{2}$ and $g(f(1)) = 2$. So, the two functions are not equal, meaning that they are not the same function.

Now, let's prove that if f and g are odd functions $f(g(x))$ is an odd function. So, we want to show that

$$f(g(-x)) = -f(g(x))$$

Let's calculate $f(g(-x))$.

$$\begin{aligned} f(g(-x)) &= f(-g(x)) \text{ since } g(-x) = -g(x) \quad [g \text{ is odd}] \\ &= -f(g(x)) \text{ since } f(-x) = -f(x) \quad [f \text{ is odd}] \end{aligned}$$

That's it.

Exercise: Prove that if $g(x)$ is an even function, then $f(g(x))$ is even regardless of what f is (even, odd, or neither).

Last Update: 10/11/11